

Matrices

An $m \times n$ matrix A is a rectangular array of mn numbers arranged in m rows and n columns.

$A =$

a_{ij} th components of A (denoted a_{ij}) is the number appearing in the i th row and j th column of A .

Example $A =$, $B =$, $C =$

An $m \times n$ matrix with all components equal zero is called the $m \times n$ zero matrix

Example $A =$

If A is $m \times n$ matrix with $m = n$ then A is called square matrix.

$A =$

$a_{11}, a_{22}, \dots, a_{nn}$ form the main diagonal of A .

Example $A =$, $B =$

A square matrix is called diagonal matrix if all terms off the main diagonal are zero.

Example $A =$, $B =$

A square matrix is called scalar matrix if all terms on the main diagonal are equal.

Example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A square matrix is called identity matrix if every terms in the main diagonal equal one.

Example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A square matrix is called upper triangular if $a_{ij} = 0$ $i > j$.

A square matrix is called lower triangular if $a_{ij} = 0$ $i < j$.

Example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ lower, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ upper

Equality of matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be $m \times n$ matrices. A is equal to B (denoted by $A = B$) if $a_{ij} = b_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$

Example $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

$A = B$ iff $x=2$, $y = 5$, $z = 0$, $w = 2$

Operation on matrices

1) Addition matrices

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be $m \times n$ matrices. The sum of A and B (denoted by $A + B$) defined to the

matrix $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, where $c_{ij} = a_{ij} + b_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$ Example $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, $A+B = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$

2) Scalar multiplication

Let $A = [a_{ij}]$ be $m \times n$ matrix and r is a real number then the scalar multiple of A by r

(denoted by rA) defined to be the matrix $D = [d_{ij}]$, where $d_{ij} = r a_{ij}$, $1 \leq i \leq m, 1 \leq j \leq n$

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $r = 2$, $2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

3) Matrix multiplication

Let $A = [a_{ij}]$ be $m \times p$ matrix, $B = [b_{ij}]$ be $p \times n$ matrix then the product of A and B (denoted by

AB) defined to be the $m \times n$ matrix $C = [c_{ij}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj}$, $1 \leq i \leq m, 1 \leq j \leq n$

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $AB = \begin{bmatrix} 10 & 16 & 23 \\ 19 & 28 & 37 \end{bmatrix}$

Let $A = [a_{ij}]$ be $m \times p$ matrix, $B = [b_{ij}]$ be $p \times n$ matrix

- 1) If $n \neq m$ then BA may not be defined

- 2) If $n = m$ then BA is $p \times p$, while AB is $m \times m$. Thus if $p \neq m$, AB and BA are different sizes.

- 3) AB and BA are both of the same size. But $AB \neq BA$.

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $AB = \begin{bmatrix} 10 & 16 & 23 \\ 19 & 28 & 37 \end{bmatrix}$, BA is not defined

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $AB = \begin{bmatrix} 10 & 16 \\ 19 & 28 \end{bmatrix}$, $BA = \begin{bmatrix} 10 & 16 \\ 19 & 28 \end{bmatrix}$

$AB \neq BA$.

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $AB = \begin{bmatrix} 10 & 16 \\ 19 & 28 \end{bmatrix}$, $BA = \begin{bmatrix} 10 & 16 \\ 19 & 28 \end{bmatrix}$

Properties of matrix addition

$$1) A+B = B + A$$

$$2) A+(B+C) = (A+B)+C$$

$$3) A+0 = 0+A$$

$$4) A+(-A) = (-A)+A = 0$$

Example $A =$, $B =$, $C =$,

$$A+B == B + A$$

$$(A+B)+C = + =$$

$$A+(B+C) = + =$$

$$-A =$$

$$A+(-A) = + =$$

$$(-A)+A = + =$$

$$A+0 = + =$$

$$0+A = +=$$

$$A-B = A +(-B)$$

Example $A =$, $B =$

$-B =$

$A - B = A + (-B) =$

Associative law for matrix multiplication

Let $A =$ be $m \times n$ matrix , $B =$ be $n \times p$ matrix and $C =$ be $p \times q$ matrix, then $A(BC) = (AB)C$

Example Let $A =$, $B =$, $C =$,

$AB =$, $(AB)C =$

$BC =$, $A(BC) =$

Distribution law for matrix multiplication

Let $A =$ be $m \times n$ matrix , $B =$ be $n \times p$ matrix and $C =$ be $n \times q$ matrix, then $A(B+C) =$

$AB+AC$

Let $A =$ be $m \times n$ matrix , $B =$ be $m \times n$ matrix and $C =$ be $n \times q$ matrix, then $(A+B)C =$

$AC+BC$

Example Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

$$B+C = \begin{pmatrix} 6 & 4 \\ 3 & 3 \end{pmatrix}, A(B+C) = \begin{pmatrix} 10 & 14 \\ 21 & 18 \end{pmatrix}$$

$$AB = \begin{pmatrix} 10 & 14 \\ 21 & 18 \end{pmatrix}, AC = \begin{pmatrix} 10 & 14 \\ 21 & 18 \end{pmatrix}, AB+AC = \begin{pmatrix} 20 & 28 \\ 42 & 36 \end{pmatrix}$$

Example Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

$$A+B = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}, (A+B)C = \begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}$$

$$AC = \begin{pmatrix} 10 & 14 \\ 21 & 18 \end{pmatrix}, BC = \begin{pmatrix} 10 & 14 \\ 21 & 18 \end{pmatrix}$$

$$AB+AC = \begin{pmatrix} 20 & 28 \\ 42 & 36 \end{pmatrix}$$

If $AB = 0$ then $A=0$, $B=0$

Example Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, $AB = \begin{pmatrix} 10 & 14 \\ 21 & 18 \end{pmatrix}$. But $A \neq 0$, $B \neq 0$

If $AB = AC$, $A \neq 0$, then $B=C$

Example Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$AB = AC = \begin{pmatrix} 10 & 14 \\ 21 & 18 \end{pmatrix}. \text{ But } B \neq C$$

The transpose of matrix

Let $A = (a_{ij})$ be $m \times n$ matrix the transpose of A (denoted by A^T)

$$A^T = (a_{ji}), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n$$

Example $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$B = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, $B^T = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $C^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Properties of transpose

1) $(A^T)^T = A$

2) $(A + B)^T = A^T + B^T$

3) $(cA)^T = cA^T$

4) $(A^T)^T = A$

A matrix A is called symmetric if $A^T = A$

Example $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

A is symmetric since $A^T = A$

B is not symmetric since $B^T \neq B$

Properties of symmetric matrix

- 1) If A is symmetric then A^T is symmetric
- 2) If A and B are symmetric then $A+B$ is symmetric
- 3) If A and B are symmetric then AB is symmetric iff $AB = BA$

A matrix A is called anti symmetric if $A^T = -A$

Example $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is anti symmetric since $A^T = -A$

Properties of anti symmetric matrix

- 1) If A is anti symmetric then A^T is anti symmetric
- 2) If A and B are anti symmetric then $A+B$ is anti symmetric
- 3) If A and B are anti symmetric then AB is anti symmetric iff $AB = -BA$

The inverse of matrix

An $n \times n$ matrix is called nonsingular (invertible) if there is $n \times n$ matrix B such that AB

$$= BA = I_n$$

The matrix B is called an inverse of A (denoted by A^{-1})

If there exists no matrix B then A is called singular (non invertible)

Example Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$, $AB = BA = I_2$

B is an inverse of A (A is non singular)

If a matrix has an inverse then the inverse is unique

Properties of inverse

1) If A is non singular then A^{-1} is non singular and $(A^{-1})^{-1} = A$

2) If A and B are non singular then AB non singular and $(AB)^{-1} = B^{-1}A^{-1}$

3) If A is non singular then $(A^{-1})^{-1} = A$

Example Find the inverse of $A =$

$=$

$=$

$= 1$, $= 0$

$= 0$, $= 1$

$A = -2$, $b = 1$, $c = 3/2$, $d = -1/2$

$=$

Gauss Jordan method

Example $A =$

[A:] =

Divided first row on 3

Multiply first row by -2 and add to second row. Multiply first row by -4 and add to third row

Divided second row on -9

Multiply second row by -6 and add to first row. Multiply second row by 23 and add to third row

Divided third row on

Multiply third row by -1 and add to first row. Multiply third row by -1/3 and add to second row =[:]

=

Determinates

If $A =$, then $= -$

Example $A =$, then $= 10 - (-12) = 22$

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then $\det A = -6$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$\det A = -6$$

There are simple method for finding determinates of 3×3 matrix = $a_{11}a_{22}a_{33} - (a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32})$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$\det A = 1(2 \cdot 9 - 3 \cdot 8) - 2(7 \cdot 9 - 3 \cdot 63) + 3(7 \cdot 8 - 4 \cdot 56) = -6$$

Properties of determinates

1) $\det A = \det A^T$

2) $\det A = -\det A^T$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$\det A = -6$$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ 6 & 12 & 18 \end{pmatrix}$

$$\det A = -6, \det B = -6$$

$$AB =$$

$$= -10 =$$

There is a rule to finding the inverse of matrix

$$A =$$

$$=$$

Example A =

$$= 1, =$$